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901 Homework 1

September 15, 2017

Problem 7

Suppose there are N different types of coupons available when buying cereal; each box contains one coupon and the collector is seeking to collect one of each in order to win a prize. After buying n boxes, what is the probability p_n that the collector has at least one of each type?

Solution: Let A_k, $1 \leq k \leq N$, denote the event that the kth coupon was obtained after buying n boxes. Therefore, we are interested in

$$P\left(\bigcap_{k=1}^{N} A_k\right) = 1 - P\left(\bigcup_{k=1}^{N} A_k^c\right)$$

where A_k^c denotes the event that the kth coupon was not obtained. By the inclusion-exclusion principle, we know

$$P\left(\bigcup_{k=1}^{N} A_k^c\right) = \sum_{k=1}^{N} P(A_k^c) - \sum_{1\leq i<j\leq N} P(A_i^c A_j^c) + \sum_{1\leq i<j<k\leq N} P(A_i^c A_j^c A_k^c) - ... + (-1)^{N-1} P(A_1^c \cdots A_N^c).$$

First note that $P(A_1^c ... A_N^c) = 0$ since we must obtain at least one coupon. Next, consider the following arguments:

$$P(A_i^c) = \left(\frac{N-1}{N}\right)^n$$

$$P(A_i^c A_j^c) = P(A_i^c \mid A_j^c) P(A_j^c) = \left(\frac{N-2}{N-1}\right)^n \left(\frac{N-1}{N}\right)^n = \left(\frac{N-2}{N}\right)^n$$

$$P(A_i^c A_j^c A_k^c) = P(A_i^c \mid A_j^c A_k^c) P(A_j^c A_k^c) = \left(\frac{N-3}{N-2}\right)^n \left(\frac{N-2}{N}\right)^n = \left(\frac{N-3}{N}\right)^n$$

and so on. Lastly, since each pair of events, say A_i or A_iA_j or $A_iA_jA_k$, etc, are all equally likely of occurring, the summations can be captured through the choose function. Therefore,

$$P\left(\bigcup_{k=1}^{N} A_k^c\right) = \binom{N}{1}\left(\frac{N-1}{N}\right)^n - \binom{N}{2}\left(\frac{N-2}{N}\right)^n + ... + (-1)^N \binom{N}{N-1}\left(\frac{N-(N-1)}{N}\right)^n$$

$$= \sum_{k=1}^{N-1} (-1)^{k+1} \binom{N}{k} \left(\frac{N-k}{N}\right)^n.$$

and hence the probability p_n that the collector has at least one of each type is

$$p_n = 1 - \sum_{k=1}^{N-1} (-1)^{k+1} \binom{N}{k} \left(\frac{N-k}{N} \right)^n .$$

Problem 16

Suppose $\mathcal{B}$ is a σ-field of subsets of Ω and suppose $Q\colon \mathcal{B} \to [0,1]$ is a set function satisfying

(a) Q is finitely additive on $\mathcal{B}$.

(b) $0 \le Q(A) \le 1$ for all $A \in \mathcal{B}$ and $Q(\Omega) = 1$.

(c) If $A_i \in \mathcal{B}$ are disjoint and $\sum_{i=1}^{\infty} A_i = \Omega$, then $\sum_{i=1}^{\infty} Q(A_i) = 1$.

Show that Q is a probability measure; that is, show Q is σ-additive.

Solution: Recall the following lemma from the notes: Suppose μ is a finitely additive set function on a σ-field $\mathcal{B}$ with $\mu(\Omega) = 1$. Then if $A_1 \supseteq A_2 \supseteq \ldots$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$ implies that $\lim_{n\to\infty} \mu(A_n) = 0$, then μ is countably additive. We will use this lemma. First, note that Q is a finitely additive set function on $\mathcal{B}$ and $Q(\Omega) = 1$. Now suppose that $A_1 \supseteq A_2 \supseteq \ldots$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$. We must show that $\lim_{n\to\infty} Q(A_n) = 0$. Note that given the set containments, we have $A_1^c \subseteq A_2^c \subseteq \ldots$ and $\bigcup_{n=1}^{\infty} A_n^c = \Omega$. To use property (c) above, consider $B_0 = \emptyset$ and for $n \ge 1$, $B_n = A_n^c \setminus A_{n-1}^c$ (which are disjoint sets). Furthermore, $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n^c = \Omega$ and so $\sum_{i=1}^{\infty} Q(B_i) = 1$, by part (c), i.e. $\lim_{n\to\infty} \sum_{i=n+1}^{\infty} Q(B_i) = 0$. Notice also that for any n,

$$\left(\bigcup_{k=1}^{\infty} B_k \right) \Big\backslash A_n^c \subseteq \bigcup_{k=n+1}^{\infty} B_k.$$

Indeed, if $x \in \left(\bigcup_{k=1}^{\infty} B_k\right) \Big\backslash A_n^c$, then $x \in \bigcup_{k=1}^{\infty} \left(A_k^c \setminus A_{k-1}^c\right)$ and $x \notin A_n^c$. Therefore, given the set containment of the A_k^c's, $x \in A_k^c \setminus A_{k-1}^c$ for some $k \ge n+1$, i.e. $x \in \bigcup_{k=n+1}^{\infty} B_k$. Combining all of this, we have

$$\begin{aligned}
\lim_{n\to\infty} Q(A_n) &= \lim_{n\to\infty} Q\big(\Omega \setminus A_n^c\big) = \lim_{n\to\infty} Q\left(\bigcup_{k=1}^{\infty} B_k \Big\backslash A_n^c \right) \\
&\le \lim_{n\to\infty} Q\left(\bigcup_{k=n+1}^{\infty} B_k \right) \\
&\le \lim_{n\to\infty} \sum_{k=n+1}^{\infty} Q(B_k) \qquad \text{by sub-additivity} \\
&= 0
\end{aligned}$$

which shows that Q is countably additive by the lemma.

Problem 19

Let $(\Omega, \mathcal{B}, P)$ be a probability space. Call a set N *null* if $N \in \mathcal{B}$ and $P(N) = 0$. Call a set $B \subset \Omega$ *negligible* if there exists a null set N such that $B \subset N$. Notice that for B to be negligible, it is not required that B be measurable. Denote the set of all negligible subsets by $\mathcal{N}$. Call $\mathcal{B}$ *complete* (with respect to P) if every negligible set is null. Suppose that $\mathcal{B}$ is not complete. Define

$$\mathcal{B}^\star := \big\{A \cup M \colon A \in \mathcal{B}, M \in \mathcal{N}\big\}.$$

(a) Show $\mathcal{B}^\star$ is a σ-field.

Solution: Clearly $\Omega \in \mathcal{B}^\star$ since $\Omega = \Omega \cup \emptyset$ and $\Omega \in \mathcal{B}$, $\emptyset \in \mathcal{N}$. Let $C \in \mathcal{B}^\star$. Then, there exists sets $A \in \mathcal{B}$ and $M \in \mathcal{N}$ such that $C = A \cup M$. Since $M \in \mathcal{N}$, there exists a set $N \in \mathcal{B}$ such that $M \subset N$ and $P(N) = 0$. Consider

$$\begin{aligned} C^c = A^c \cap M^c &= A^c \cap \Big(N^c \cup \big(M^c \setminus N^c\big)\Big) \\ &= \big(A^c \cap N^c\big) \cup \Big(A^c \cap \big(M^c \setminus N^c\big)\Big). \end{aligned}$$

We know that $A^c \cap N^c \in \mathcal{B}$. Notice that $A^c \cap (M^c \setminus N^c) \subset M^c \setminus N^c \subset \Omega \setminus N^c$, and since $N^c \subset \Omega$, $P(\Omega \setminus N^c) = P(\Omega) - P(N^c) = 0$, i.e. $\Omega \setminus N^c$ is a null set. Therefore, $A^c \cap (M^c \setminus N^c) \in \mathcal{N}$, and so $C^c \in \mathcal{B}^\star$. Now suppose $C_1, C_2, ... \in \mathcal{B}^\star$. Then there exists $A_1, A_2, ... \in \mathcal{B}$ and $M_1, M_2, ... \in \mathcal{N}$ such that $C_n = A_n \cup M_n$. Note

$$\bigcup_{n=1}^{\infty} C_n = \left(\bigcup_{n=1}^{\infty} A_n\right) \bigcup \left(\bigcup_{n=1}^{\infty} M_n\right).$$

Clearly $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$ since $\mathcal{B}$ is a σ-field. Also, since $M_n \in \mathcal{N}$, there exists a null set $O_n \in \mathcal{B}$ such that $M_n \subset O_n$ and $P(O_n) = 0$. Therefore, $\bigcup_{n=1}^{\infty} M_n \subset \bigcup_{n=1}^{\infty} O_n$ and by subadditivity

$$P\left(\bigcup_{n=1}^{\infty} O_n\right) \le \sum_{n=1}^{\infty} P(O_n) = 0,$$

i.e. $\bigcup_{n=1}^{\infty} O_n \in \mathcal{B}$ and is null. Therefore, $\bigcup_{n=1}^{\infty} M_n \in \mathcal{N}$. This shows that $\bigcup_{n=1}^{\infty} C_n \in \mathcal{B}^\star$ and hence $\mathcal{B}^\star$ is a σ-field.

(b) Show that if $A_i \in \mathcal{B}$ and $M_i \in \mathcal{N}$ for $i = 1, 2$ and

$$A_1 \cup M_1 = A_2 \cup M_2,$$

then $P(A_1) = P(A_2)$.

Solution: Notice that $A_1 \cup M_1 = A_2 \cup M_2$ implies that $\big(A_1 \cup M_1\big) \setminus M_2 \subset A_2$, and so we know $A_1 \setminus M_2 \subset A_2$. Since $M_2 \in \mathcal{N}$, there exists a null set $B_2 \in \mathcal{B}$ such that $M_2 \subset B_2$ and $P(B_2) = 0$. Therefore, we have $A_1 \setminus B_2 \subset A_1 \setminus M_2 \subset A_2$. Now, consider by inclusion-exclusion

$$P(A_1 \setminus B_2) = P(A_1 \cap B_2^c) = P(A_1) + P(B_2^c) - P(A_1 \cup B_2^c) = P(A_1) + 1 - 1 = P(A_1).$$

Therefore, we have $P(A_1) = P(A_1 \setminus B_2) \le P(A_2)$. Reversing the first step to $A_1 \supset \big(A_2 \cup M_2\big) \setminus M_1$ gives the result.

(c) Define $P^\star\colon \mathcal{B}^\star \to [0,1]$ by

$$P^\star(A \cup M) = P(A), \quad A \in \mathcal{B}, M \in \mathcal{N}.$$

Show that $P^\star$ is an extension of P to $\mathcal{B}^\star$.

Solution: Let $B \in \mathcal{B}$. Then, $P^\star(B) = P^\star(B \cup \emptyset) = P(B)$. This shows $P^\star$ is an extension of P to $\mathcal{B}^\star$.

(d) If $B \subset \Omega$ and $A_i \in \mathcal{B}, i = 1, 2$ and $A_1 \subset B \subset A_2$ and $P(A_2 \setminus A_1) = 0$, then show $B \in \mathcal{B}^\star$.

Solution: Notice that $B = A_1 \cup \big(B \setminus A_1\big)$. Also $B \setminus A_1 \subset A_2 \setminus A_1$ and $A_2 \setminus A_1$ is null. Therefore, $B \in \mathcal{B}^\star$.

(e) Show $\mathcal{B}^\star$ is complete. Thus every σ-field has a completion.

Solution: Let $A \in \mathcal{N}$. Then by definition of $P^\star$,

$$P^\star(\emptyset \cup A) = P(\emptyset) = 0.$$

Since $A = \emptyset \cup A$, we have A is a null set in $\mathcal{B}^\star$. Therefore, $\mathcal{B}^\star$ is complete.

(f) Suppose $\Omega = \mathbb{R}$ and $\mathcal{B} = \mathcal{B}(\mathbb{R})$. Let $p_k \geq 0$ and $\sum_k p_k = 1$. Let $\{a_k\}$ be any sequence in $\mathbb{R}$. Define P by

$$P\big(\{a_k\}\big) = p_k, \qquad P(A) = \sum_{a_k \in A} p_k, \qquad A \in \mathcal{B}.$$

What is the completion of $\mathcal{B}$?

Solution: Let $A \subset \mathbb{R}$. Define the following two sets: $G = \big\{a_k \colon P\big(\{a_k\}\big) > 0\big\}$ and $B = \big\{a_k \in A \colon P\big(\{a_k\}\big) > 0\big\}$. First notice that $P(G) = 1$ and also

$$A = B \cup \big(A \setminus B\big).$$

Clearly $B \in \mathcal{B}$ since it can be written as a countable union of measurable sets $\{a_k\}$. Also, since $A \setminus B \subset G^c$ and $P(G^c) = 0$, we have $A \setminus B$ is a negligible set. Therefore, A is in the completion of $\mathcal{B}$. Since the set $A \subset \mathbb{R}$ was arbitrary, this implies that the completion of $\mathcal{B}$ must be the power set of $\mathbb{R}$, i.e. $\mathcal{P}(\mathbb{R})$.

(g) Say that the probability space $(\Omega, \mathcal{B}, P)$ has a complete extension $(\Omega, \mathcal{B}_1, P_1)$ if $\mathcal{B} \subset \mathcal{B}_1$ and $P_1|_{\mathcal{B}} = P$. The previous problem (c) showed that every probability space has a complete extension. However, this extension may not be unique. Suppose that $(\Omega, \mathcal{B}_2, P_2)$ is a second complete extension of $(\Omega, \mathcal{B}, P)$. Show that P_1 and P_2 may not agree on $\mathcal{B}_1 \cap \mathcal{B}_2$.

Solution: Let $\Omega = \{1,2,3\}$ and consider the σ-field $\mathcal{B} = \big\{\emptyset, \{1\}, \{2,3\}, \Omega\big\}$ where $P(\{1\}) = 1/2$ and $P(\{2,3\}) = 1/2$. Now, define the extensions $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{P}(\Omega)$, which are both complete because the only negligible set is the empty set which is trivially null. However, if we define the probabilities

$$\begin{aligned} P_1(\{1\}) &= 1/2 & P_2(\{1\}) &= 1/2 \\ P_1(\{2\}) &= 1/4 & P_2(\{2\}) &= 1/3 \\ P_1(\{3\}) &= 1/4 & P_2(\{3\}) &= 1/6 \end{aligned}$$

we see that P_1 and P_2 do not agree on $\mathcal{B}_1 \cap \mathcal{B}_2$ since, say, $P_1(\{2\}) \neq P_2(\{2\})$. Note that these values were chosen so that $P(\{1\}) = P_1(\{1\}) = P_2(\{1\})$ and also $P(\{2,3\}) = P_1(\{2,3\}) = P_2(\{2,3\})$, where $P_i(\{2,3\}) = P_i(\{2\}) + P_i(\{3\})$ for $i = 1, 2$.

(h) Is there a *minimal* extension?

Solution: We will show that $(\Omega, \mathcal{B}^\star, P^\star)$ is the minimal complete extension. Let $(\Omega, \mathcal{B}', P')$ be another complete extension of $(\Omega, \mathcal{B}, P)$. Then any set $M \in \mathcal{N}$ is a null set in $\mathcal{B}'$, i.e. $\mathcal{N} \subset \mathcal{B}'$. Also, clearly $\mathcal{B} \subset \mathcal{B}'$ since $\mathcal{B}'$ is an extension. Now, take any $B \in \mathcal{B}^\star$, i.e. $B = A \cup M$ where $A \in \mathcal{B}$ and $M \in \mathcal{N}$. Since $\mathcal{B} \subset \mathcal{B}'$ and $\mathcal{N} \subset \mathcal{B}'$, we have $A, M \in \mathcal{B}'$ and so too is $A \cup M$. Thus, $B \in \mathcal{B}'$ and we conclude that $\mathcal{B}^\star \subset \mathcal{B}'$. Therefore, $(\Omega, \mathcal{B}^\star, P^\star)$ is the minimal extension.